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T1 vs. T2: On the Definition of Mixed Strategies in Noncooperative Games

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Abstract. This paper identifies a central role of the topological separation axiom T_1 in the definition of mixed strategies in noncooperative games with arbitrary pure strategy spaces. Our main result says that a pure strategy space is topologically T_1 if and only if (i) all singleton strategy sets are Borel, (ii) all Dirac measures are regular, and (iii) the canonical mapping from pure strategies to Dirac measures is one-to-one. The analysis therefore suggests that the T_1 separation axiom is a minimum requirement on the topology of a pure strategy space when randomization is allowed for. Using an example, we show that the T_1 assumption is indeed missing from the minimax theorem of Mertens (1986).

Keywords. Mixed strategies, Hausdorff spaces, T_1 separation axiom, minimax theorem

JEL classification. C72: Noncooperative Games

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1 Introduction

Since the path-breaking contributions of Glicksberg (1952) and Fan (1953), it has been standard to define a mixed strategy as a regular probability measure on the Borel sets of the underlying pure strategy space.[1] Further, it has been very common to assume that the topology on the underlying pure strategy spaces is Hausdorff. However, the rationale for using the Hausdorff separation axiom, except maybe that it naturally holds in metrizable spaces, has remained less clear. For example, it is known from Sion (1958) and Reny (1999) that neither the existence of a value in a two-person zero-sum game nor the existence of pure-strategy Nash equilibrium (PSNE) in a noncooperative game hinges on any separation axiom.[2] It is therefore remarkable that, with the sole exception of Mertens (1986), the Hausdorff assumption has been imposed in virtually every analysis of mixed-strategy Nash equilibria (MSNE).

In this paper, we consider noncooperative games in which pure strategy spaces are arbitrary topological spaces, and explore the role of the T_1 separation axiom for the definition of mixed strategies. Imposed on a pure strategy set X equipped with some topology, the T_1 separation axiom says that, for any pure strategy $x \in X$, the singleton set $\{x\}$ is a closed set. The T_1 separation axiom is necessarily satisfied in all Hausdorff (i.e., T_2) strategy spaces, but conversely, T_1-spaces need not be Hausdorff. Our main result says that the topology on a player's pure strategy space satisfies the T_1 separation axiom if and only if (i) all singletons are Borel measurable, (ii) all Dirac measures are regular, and (iii) the canonical mapping that transforms any given pure strategy into the corresponding Dirac

[1] See, however, adaptions of the concept of mixed strategies in extensive games (Kuhn, 1953; Aumann, 1964) and Bayesian games (Milgrom and Weber, 1985).

[2] Similarly, Balder (1999, App. A) extended Kakutani's fixed point theorem to non-Hausdorff spaces. See also more the recent work by Goubault-Larrecq (2018) and Khan et al. (2024).

measure is injective. This equivalence suggests that the T_1 separation axiom is a minimum restriction for any pure strategy spaces over which randomized choices are admitted. As an application, we provide an example of a strategy space that does not satisfy the T_1 separation axiom and for which all finitely supported strategies are irregular. This example shows, in particular, that the T_1 assumption is missing from the non-Hausdorff minimax theorem in Mertens (1986, Thm. 3).

We start with a motivating example illustrating the role of topological separation properties for the definition of mixed strategies. For convenience, the example works with a finite normal form game, but the discussion captures a general problem.

Example 1. *Consider a Prisoner's Dilemma game and suppose that the pure strategy spaces, $X = Y = \{C, D\}$, are equipped with the* Sierpiński *topology $\mathcal{T} = \{\emptyset, \{D\}, \{C, D\}\}$. Then, the regularity property of any mixed strategy μ implies*

$$\begin{aligned} \mu(\{C\}) &= \inf\{\mu(U) : U \supseteq \{C\} \textit{ open}\} \\ &= \mu(\{C, D\}) \\ &= 1. \end{aligned}$$

Hence, any mixed strategy satisfies $\mu(\{C\}) = 1$. In fact, the unique MSNE outcome is (C, C).

Thus, in the absence of any separation axiom, one can define a topology on the pure strategy spaces of a Prisoner's Dilemma game such that cooperation becomes the unique MSNE outcome. To avoid such pathological cases, it seems desirable to focus on games for which the mixed extension *embed pure strategies*, in the sense that (i) all singleton strategy sets are measurable, (ii) all Dirac measures are regular, and (iii) the mapping from pure strategies to corresponding Dirac mixed strategies is one-to-one. We then establish that the T_1 separation axiom is equivalent to these three conditions. Thus, for obtaining a meaningful mixed extension, imposing T_1 on the pure strategy space seem essential.

A useful side-effect of the T_1 separation axiom is that the mixed extension is nonempty. The salience of this observation becomes clear when we apply our results to the non-Hausdorff minimax theorem derived by Mertens (1986). Specifically, we will provide an example of a strategy space that does not satisfy the T_1 separation axiom and for which all convex mixtures of Dirac measures are irregular. This shows, in particular, that the T_1 separation axiom is indispensible for any *mixed-strategy* minimax theorem, e.g., one that claims the existence of, e.g., an ε-equilibrium in finitely supported strategies. In contrast, no separation axiom is needed to derive a *pure-strategy* minimax theorem, as illustrated by the work of Goubault-Larrecq (2018).

The remainder of this paper is organized as follows. Section 2 concerns preliminaries. Section 3 contains our main result. Section 4 discusses implications for the analysis of Mertens (1986). Section 5 concludes.

2 Preliminaries

This section prepares the analysis by reviewing the necessary background on separation axioms (Subsection 2.1) and regular probability measures (Subsection 2.2).

2.1 Separation Axioms in General Topology

A topological space[3] $(X, \mathcal{T})$ is *Hausdorff* (is a T_2-*space*, or satisfies the T_2 *separation axiom*) if, for any two distinct points $x, x' \in X$, there are disjoint sets $U, U' \in \mathcal{T}$ such that $x \in U$ and $x' \in U'$. The Hausdorff property is equivalent to the condition that the limit of any convergent net is unique (Kelley, 1975, Thm. 2.3). In Hausdorff spaces, compact sets are closed. Conversely, however, the property that compact sets are closed is in general strictly weaker than the Hausdorff property even for a compact space (Wilansky, 1967, Thm. 1).

A topological space is called a T_1-*space* (or satisfies the T_1 *separation axiom*) if any singleton is closed (Kelley, 1975, p. 56). The following lemma provides an alternative way to think about the T_1 separation axiom.

Lemma 1. *A topological space $(X, \mathcal{T})$ satisfies the T_1 separation axiom if and only if, for any two distinct points $x, x' \in X$, there exists $U \in \mathcal{T}$ such that $x \in U$ and $x' \notin U$.*

Proof. See Engelking (1977, Sec. 1.5). □

In particular, any Hausdorff space is T_1, but the reverse is not generally true.[4]

[3] A *topology* $\mathcal{T}$ on X is a collection of *open* subsets of X such that (i) $\emptyset$ and X are open, (ii) the union of any collection of open sets is open, and (iii) the intersection of any finite collection of open is open. A *topological space* $(X, \mathcal{T})$ consists of a set X and a topology $\mathcal{T}$ on X. A set is *closed* in X if its complement is open.

[4] For further discussion of T_1-spaces, see Reilly (1995) and Clontz (2024).

2.2 Regular Probability Measures

Given a measurable space,[5] a *probability measure* on X is a countably additive, nonnegative set function $\mu : \Sigma \to \mathbb{R}$ such that $\mu(X) = 1$. The *support* of a probability measure μ is the intersection of all closed sets $S \subseteq X$ such that $\mu(S) = 1$. For $x \in X$, the *Dirac* measure δ_x is given by $\delta_x(S) = 0$ if $x \notin S$ and $\delta_x(S) = 1$ if $x \in S$, for any $S \in \mathcal{B}(X)$.

Given a topological space $(X, \mathcal{T})$, the elements of the smallest σ-algebra $\mathcal{B}(X)$ containing all open sets are called the *Borel sets*. A probability measure μ defined on the Borel sets of X is *regular* if for each $S \in \mathcal{B}(X)$ and $\varepsilon > 0$, there is a closed set K and an open set U such that $K \subseteq S \subseteq U$ and $\mu(U \backslash K) < \varepsilon$ (Dunford and Schwartz, 1958, Sec. III.5).

3 Pure and Mixed Strategies

The definition of the mixed strategies does not impose any separation axiom on the pure strategy space. As we have seen in Example 1, however, with no separation axiom imposed at all, strange things may happen. The following is the main result of the present paper.

Proposition 1. *A pure strategy space X is a T_1-space if and only if the following three conditions are satisfied:*

(i) Any singleton is Borel measurable;

(ii) any Dirac measure δ_x is regular;

[5] A *σ-algebra* Σ on a set X is a collection of subsets of X such that (i) $X \in \Sigma$, (ii) if $A \in \Sigma$, then $A^c = X \setminus A \in \Sigma$, and (iii) if $\{A_n\}_{n=1}^{\infty}$ is a countable collection of sets in Σ, then $\bigcup_{n=1}^{\infty} A_n \in \Sigma$. A *measurable space* is a pair (X, Σ), where X is a nonempty set and Σ is a σ-algebra on X. Given a measurable space (X, Σ), a *set function* is a mapping $\mu : \Sigma \to \mathbb{R}$. A set function is *nonnegative* if it does not attain negative values. A set function is *countably additive* if, for any countable collection $\{S_n\}_{n=1}^{\infty}$ of pairwise disjoint sets in Σ, we have $\mu\left(\bigcup_{n=1}^{\infty} S_n\right) = \sum_{n=1}^{\infty} \mu(S_n)$.

(iii) the mapping $x \mapsto \delta_x$ is one-to-one.

Proof. We prove each direction separately.

(Only if) Suppose that X is a T_1-space, and take any $x \in X$. Then, $\{x\}$ is closed and, hence, Borel, proving property (i). Next, we will show that the Dirac measure δ_x is regular. By definition, δ_x is regular if for each $S \in \mathcal{B}(X)$ and $\varepsilon > 0$, there is a closed set $K \subseteq X$ and an open set $U \subseteq X$ such that $\delta_x(U \backslash K) < \varepsilon$. Take a Borel set $S \subseteq X$ and some $\varepsilon > 0$. There are two cases. Suppose first that $x \notin S$. Then, we choose the closed set $K = \emptyset \subseteq S$ and the open set $U = X \backslash \{x\} \supseteq S$ and note that $\delta_x(U \backslash K) = \delta_x(X \backslash \{x\}) = 0 < \varepsilon$. Suppose next that $x \in S$. Then, we choose the closed set $K = \{x\} \subseteq S$ and the open set $U = X \supseteq S$, noting that $\delta_x(U \backslash K) = \delta_x(X \backslash \{x\}) = 0 < \varepsilon$. In sum, we have shown that, indeed, δ_x is regular, so that property (ii) has been verified as well. Property (iii) is now immediate because the Dirac measure can be tested on singletons as a consequence of property (i).

(If) By contradiction. Suppose that conditions (i) through (iii) in the statement of the proposition hold, but the pure strategy space X is not T_1. Then, by Lemma 1 above, there exist $x, x' \in X$ such that $x \neq x'$ and such that any open set $U \ni x$ contains also x'. By assumption, the singleton $\{x\}$ is Borel. Any closed set K contained in $\{x\}$ is either the empty set or $\{x\}$. Moreover, any open set U that contains $\{x\}$ as a subset necessarily contains also x'. Therefore, regardless of the choice of U and K, the difference set $U \backslash K$ contains x'. Hence, $\delta_{x'}(U \backslash K) = 1$, in conflict with the regularity of $\delta_{x'}$. □

Proposition 1 suggests that the T_1 separation axiom is a desirable condition on the topology of a pure strategy space because it ensures the availability of all pure strategies in the mixed extension. Notably, this also guarantees that mixed strategy spaces are nonempty, which would otherwise not be self-evident.

Corollary 1. *Suppose that X is nonempty and satisfies the T_1 separation axiom. Then, the set of mixed strategies on X is nonempty.*

Proof. Immediate from Proposition 1. □

Thus, the T_1 separation axiom captures the idea that a player has access to her pure strategies in the mixed extension.

4 Implications for Mertens (1986)

We have seen above that the T_1 separation axiom ensures that any singleton is Borel and that any Dirac measure is regular. As finite sums of regular measures are regular, it follows that, in a T_1-space, any convex combination of Dirac measures,

$$\mu_i = \sum_{m=1}^{M} \lambda_m \delta_{x^m},$$

with mass points $x^1, \ldots, x^M \in X$ and probability weights $\lambda_1, \ldots, \lambda_M \geq 0$ such that $\sum_{m=1}^{M} \lambda_m = 1$, is regular. Thus, the T_1 separation axiom ensures that all convex combinations of Dirac measures qualify as mixed strategies. Moreover, the support of any such strategy is closed in that case because the T_1 separation axiom ensures that every singleton is closed. Therefore, any convex combination of Dirac measures is finitely supported in a T_1-space. In the absence of the T_1 separation axiom, however, there may be no finitely supported mixed strategies at all, as the following example illustrates.

Example 2. *Consider the symmetric two-person zero-sum game, where pure strategy spaces $X = Y = [0, 1]$ are equipped with the* indiscrete *topology $\mathcal{T}^{\#} = \{\emptyset, X\}$, and payoff function $u = 0$, i.e., constant zero. Then, each player has* *precisely one mixed strategy at her disposition, namely μ given by $\mu(\emptyset) = 0$ and $\mu(X) = 1$.*

This indiscrete topology is the coarsest topology of all, and does not even satisfy the T_1 separation axiom. Notably, since the unique regular Borel probability measure has the infinite set X as its support, there are no finitely supported mixed strategies. As discussed already in the Introduction, the example above is of some relevance for the non-Hausdorff minimax theorem of Mertens (1986, Thm. 3), which states the existence of an ε-equilibrium in finitely supported strategies. Namely, in the example constructed above, there are no finitely supported mixed strategies, which implies that the T_1 assumption is missing in the statement of the minimax theorem.

5 Conclusion

While mixed strategies have been commonly defined over Hausdorff strategy spaces, the present analysis has shown that the more flexible T_1 separation axiom ("points are closed") should be the more natural requirement. This axiom ensures that a player, when allowed to randomize, does not lose access to any of her pure strategies.

The reader will have noted that we did not impose any restriction on the number of players. Therefore, all we have said applies equally to a Savage-style single-agent framework, where a decision maker chooses lotteries over acts. It also applies without change to noncooperative games with infinitely many players. Indeed, our comments concerned exclusively the relationship between pure and randomized choices or, as the matter may be, between pure states and beliefs.

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